Optimal Homologous Cycles, Total Unimodularity, and Linear Programming*

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Abstract

Given a simplicial complex with weights on its simplices, and a nontrivial cycle on it, we are interested in finding the cycle with minimal weight which is homologous to the given one. Assuming that the homology is defined with integer ($\mathbb{Z}$) coefficients, we show the following (Theorem 5.2):

> *For a finite simplicial complex K of dimension greater than p, the boundary matrix $[\partial_{p+1}]$ is totally unimodular if and only if $H_p(L, L_0)$ is torsion-free, for all pure subcomplexes L_0, L in K of dimensions p and $p+1$ respectively, where $L_0 \subset L$.*

Because of the total unimodularity of the boundary matrix, we can solve the optimization problem, which is inherently an integer programming problem, as a linear program and obtain an integer solution. Thus the problem of finding optimal cycles in a given homology class can be solved in polynomial time. This result is surprising in the backdrop of a recent result which says that the problem is NP-hard under $\mathbb{Z}_2$ coefficients which, being a *field*, is in general easier to deal with. Our result implies, among other things, that one can compute in polynomial time an optimal $(d-1)$-cycle in a given homology class for any triangulation of an orientable compact d-manifold or for any finite simplicial complex embedded in $\mathbb{R}^d$. Our optimization approach can also be used for various related problems, such as finding an optimal *chain* homologous to a given one when these are not cycles. Our result can also be viewed as providing a topological characterization of total unimodularity.

1 Introduction

Topological cycles in shapes embody their important features. As a result they find applications in scientific studies and engineering developments. A version of the problem that often appears in practice is that given a cycle in the shape, compute the shortest cycle in the same topological class (homologous). For example, one may generate a set of cycles from a simplicial complex using the persistence algorithm [10] and then ask to tighten them while maintaining their homology classes. For two dimensional surfaces, this problem and its relatives have been widely studied in recent years; see, for example, [2, 3, 5, 6, 8]. A natural question is to consider higher dimensional spaces which allow higher dimensional cycles such as closed surfaces within a three dimensional topological space. High dimensional applications arise, for example, in the modeling of sensor networks by Vietoris-Rips complexes of arbitrary dimension [7, 20]. Not surprisingly, these generalizations are hard to compute which is confirmed by a recent result of Chen and Freedman [4]. Notwithstanding this negative development, our result shows that optimal homologous cycles in any finite

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dimension are polynomial time computable for a large class of shapes if homology is defined with integer coefficients.

Let K be a simplicial complex. Informally, a p-cycle in K is a collection of p-simplices whose boundaries cancel mutually. One may assign a non-zero weight to each p-simplex in K which induces a weighted 1-norm for each p-cycle in K. For example, the weight of a p-simplex could be its volume. Given any p-cycle c in K, our problem is to compute a p-cycle c^* which has the minimal weighted 1-norm in the homology class of c. If some of the weights are zero the problem can still be posed and solved, except that one may not call it weighted 1-norm minimization. The homology classes are defined with respect to coefficients in an abelian group such as $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_n$ etc. Often, the group $\mathbb{Z}_2$ is used mainly because of simplicity and intuitive geometric interpretations.

Chen and Freedman [4] show that under $\mathbb{Z}_2$ coefficients, computing an optimal p-cycle c^* is NP-hard for $p \geq 1$. Moreover, their result implies that various relaxations may still be NP-hard. For example, computing a constant factor approximation of c^* is NP-hard. Even if the rank of the p-dimensional homology group is constant, computing c^* remains NP-hard for $p \geq 2$. The only settled positive case is a result of Chambers, Erickson, and Nayyeri [3] who show that computing optimal homologous loops for surfaces with constant genus is polynomial time solvable though they prove the problem is NP-hard if the genus is not a constant.

The above negative results put a roadblock in computing optimal homologous cycles in high dimensions. Fortunately, our result shows that it is not so hopeless – if we switch to the coefficient group $\mathbb{Z}$ instead of $\mathbb{Z}_2$, the problem becomes polynomial time solvable for a fairly large class of spaces. This is a little surprising given that $\mathbb{Z}$ is not a *field* and so seems harder to deal with than $\mathbb{Z}_2$ in general. For example, $\mathbb{Z}_2$-valued chains form a vector space, but $\mathbb{Z}$-valued chains do not.

The problem of computing an optimal homologous cycle (or more generally, chain) can be cast as a linear optimization problem. Consequently, the problem becomes polynomial time solvable if the homology group is defined over the reals, since it can be solved by linear programming. Indeed this is the approach taken by Tahbaz-Salehi and Jadbabaie [20]. However, in general the optimal cycle in that case may have fractional coefficients for its simplices, which may be awkward in certain applications. One advantage of using $\mathbb{Z}$ is that simplices appear with integral coefficients in the solution. On the other hand, the linear programming has to be replaced by integer programming in the case of $\mathbb{Z}$. Thus, it is not immediately clear if the optimization problem is polynomial time solvable. One issue in accommodating $\mathbb{Z}$ as the coefficient ring is that integral coefficients other than 0, 1, or -1 do not have natural geometric meaning. Nevertheless, our experiments suggest that optimal solutions in practice may contain coefficients only in $\{-1, 0, 1\}$. Furthermore, as we show later, we can put a constraint in our optimization to enforce the solution to have coefficients in $\{-1, 0, 1\}$.

Our main observation is that the optimization problem that we formulate can be solved by linear programming under certain conditions, although it is inherently an integer programming problem. It is known that a linear program provides an integer solution if and only if the constraint matrix has a property called *total unimodularity*. A matrix is totally unimodular if and only if each of its square submatrices has a determinant of 0, 1, or -1. We give a precise topological characterization of the complexes for which the constraint matrix is totally unimodular. For this class of complexes the optimal cycle can be computed in time polynomial in the number of simplices. Totally unimodular matrices have a well-known geometric characterization – that the corresponding constraint polyhedron is integral [15, Theorem 19.1]. Our result provides a topological characterization as well.

We can allow several variations to our problem because of our optimization based approach. For example, we can probe into intermediate solutions; we can produce the chain that bounds the difference of the input and optimal cycles, and so forth. In fact, we can also find an optimal chain homologous to a given one when the chains are not cycles. In other words, we can leverage the flexibility of the optimization formulation by linking results from two apparently different fields, optimization theory and algebraic topology.

2 Background

Since our result bridges the two very different fields of algebraic topology and optimization, we recall some relevant basic concepts and definitions from these two fields.

2.1 Basic definitions from algebraic topology

Let K be a finite simplicial complex of dimension greater than p. A p-chain with $\mathbb{Z}$ coefficients in K is a *formal sum* of a set of oriented p-simplices in K where the sum is defined by addition in $\mathbb{Z}$. Equivalently, it is an integer valued function on the oriented p-simplices, which changes sign when the orientation is reversed [14, page 37].

Two p-chains can be added by adding their values on corresponding p-simplices, resulting in a group $C_p(K)$ called the p-chain group of K. The *elementary chain basis* for $C_p(K)$ is the one consisting of integer valued functions that take the value 1 on a single oriented p-simplex, -1 on the oppositely oriented simplex, and 0 everywhere else. For an oriented p-simplex σ, we use σ to denote both the simplex and the corresponding elementary chain basis element. The group $C_p(K)$ is free and abelian. The boundary of an oriented p-simplex $\sigma = [v_0, \dots, v_p]$ is given by

$$\partial_p \, \sigma = \sum_{i=0}^{p} (-1)^i [v_0, .., \widehat{v_i}, .., v_p] \, ,$$

where $\widehat{v_i}$ denotes that the vertex v_i is to be deleted. This function on p-simplices extends uniquely [14, page 28] to the *boundary operator* which is a homomorphism:

$$\partial_p \colon C_p(K) \to C_{p-1}(K) \, .$$

Like a linear operator between vector spaces, a homomorphism between free abelian groups has a unique matrix representation with respect to a choice of bases [14, page 55]. The matrix form of ∂_p will be denoted $[\partial_p]$. Let $\{\sigma_i\}_{i=0}^{m-1}$ and $\{\tau_j\}_{j=0}^{n-1}$ be the sets of oriented $(p-1)$- and p-simplices respectively in K, ordered arbitrarily. Thus $\{\sigma_i\}$ and $\{\tau_j\}$ also represent the elementary chain bases for $C_{p-1}(K)$ and $C_p(K)$ respectively. With respect to such bases $[\partial_p]$ is an $m \times n$ matrix with entries 0, 1 or -1. The coefficients of $\partial_p \, \tau_j$ in the $C_{p-1}(K)$ basis become the column j (counting from 0) of $[\partial_p]$.

The kernel $\ker \partial_p$ is called the group of *p-cycles* and denoted $Z_p(K)$. The image $\operatorname{im} \partial_{p+1}$ forms the group of *p-boundaries* and denoted $B_p(K)$. Both $Z_p(K)$ and $B_p(K)$ are subgroups of $C_p(K)$. Since $\partial_p \circ \partial_{p+1} = 0$, we have that $B_p(K) \subseteq Z_p(K)$, that is, all p-boundaries are p-cycles though the converse is not necessarily true. The p dimensional homology group is the quotient group $H_p(K) = Z_p(K)/B_p(K)$. Two p-chains c and c' in K are *homologous* if $c = c' + \partial_{p+1} \, d$ for some $(p+1)$-chain d in K. In particular, if $c = \partial_{p+1} \, d$, we say c is homologous to zero. If a cycle c is not homologous to zero, we call it a *non-trivial cycle*.

For a finite simplicial complex K, the groups of chains $C_p(K)$, cycles $Z_p(K)$, and $H_p(K)$ are all finitely generated abelian groups. By the fundamental theorem of finitely generated abelian groups [14, page 24] any such group G can be written as a direct sum of two groups $G = F \oplus T$ where $F \cong (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})$ and $T \cong (\mathbb{Z}/t_1 \oplus \cdots \oplus \mathbb{Z}/t_k)$ with $t_i > 1$ and t_i dividing t_{i+1}. The subgroup T is called the *torsion* of G. If $T = 0$, we say G is *torsion-free*.

Let L_0 be a subcomplex of a simplicial complex L. The quotient group $C_p(L)/C_p(L_0)$ is called the group of *relative chains* of L modulo L_0 and is denoted $C_p(L, L_0)$. The boundary operator $\partial_p \colon C_p(L) \to C_{p-1}(L)$ and its restriction to L_0 induce a homomorphism

$$\partial_p^{(L,L_0)} \colon C_p(L, L_0) \to C_{p-1}(L, L_0) \, .$$

As before, we have $\partial_p^{(L,L_0)} \circ \partial_{p+1}^{(L,L_0)} = 0$. Writing $Z_p(L, L_0) = \ker \partial_p^{(L,L_0)}$ for *relative cycles* and $B_p(L, L_0) = \operatorname{im} \partial_{p+1}^{(L,L_0)}$ for *relative boundaries*, we obtain the *relative homology group* $H_p(L, L_0) = Z_p(L, L_0)/B_p(L, L_0)$. Sometimes, to distinguish it from relative homology, the usual homology $H_p(L)$ is called the *absolute homology group* of L.

2.2 Total unimodularity and optimization

Recall that a matrix is *totally unimodular* if the determinant of each square submatrix is 0, 1, or -1. The significance of total unimodularity in our setting is due to the following result:

Theorem 2.1. [24] *Let A be an $m \times n$ totally unimodular matrix and $\mathbf{b}$ an integral vector, i.e., $\mathbf{b} \in \mathbb{Z}^m$. Then the polyhedron $\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ is integral, meaning that $\mathcal{P}$ is the convex hull of the integral vectors contained in $\mathcal{P}$. In particular, the extreme points (vertices) of $\mathcal{P}$ are integral. Similarly the polyhedron $\mathcal{Q} := \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \geq \mathbf{b}\}$ is integral.*

The following corollary shows why the above result is significant for optimization problems. Consider an integral vector $\mathbf{b} \in \mathbb{Z}^m$ and a real vector of cost coefficients $\mathbf{f} \in \mathbb{R}^n$. Consider the *integer* linear program

$$\min\ \mathbf{f}^T\mathbf{x} \quad \text{subject to} \quad A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{x} \in \mathbb{Z}^n\,. \tag{1}$$

Corollary 2.2. *Let A be a totally unimodular matrix. Then the integer linear program* (1) *can be solved in time polynomial in the dimensions of A.*

Proof. Relax the integer linear program (1) to a linear program by removing the integrality constraint $\mathbf{x} \in \mathbb{Z}^n$. Then an interior point method for solving linear programs will find a real solution $\mathbf{x}^*$ in polynomial time [15] if it exists, and indicates the unboundedness or infeasibility of the linear program otherwise. In fact, since the matrix A has entries 0, 1 or -1, one can solve the linear program in strongly polynomial time [21, 22]. That is, the number of arithmetic operations do not depend on $\mathbf{b}$ and $\mathbf{f}$ and solely depends on the dimension of A. One still needs to show that the solution $\mathbf{x}^*$ is integral.

If the solution is unique then it lies at a vertex of the polyhedron $\mathcal{P}$ and thus it will be integral because of Theorem 2.1. If the optimal solution set is a face of $\mathcal{P}$ which is not a vertex then an interior point method may at first find a non-integral solution. However, by [1, Corollary 2.2] the polyhedron $\mathcal{P}$ must have at least one vertex. Then, by [1, Theorem 2.8] if the optimal cost is finite, there exists a vertex of $\mathcal{P}$ where that optimal cost is achieved. Following the procedure described in [12], starting from the possibly non-integral solution obtained by an interior point method one can find such an integral optimal solution at a vertex in polynomial time. $\square$

3 Problem formulation

Let K be a finite simplicial complex of dimension p or more. Given an integer valued p-chain $x = \sum_{i=0}^{m-1} x(\sigma_i)\,\sigma_i$ we use $\mathbf{x} \in \mathbb{Z}^m$ to denote the vector formed by the coefficients $x(\sigma_i)$. Thus, $\mathbf{x}$ is the representation of the chain x in the elementary p-chain basis, and we will use $\mathbf{x}$ and x interchangeably. For a vector $\mathbf{v} \in \mathbb{R}^m$, the *1-norm* (or ℓ^1-norm) $\|\mathbf{v}\|_1$ is defined to be $\sum_i |v_i|$. Let W be any real $m \times m$ diagonal matrix with diagonal entries w_i. Then, the 1-norm of $W\,\mathbf{v}$, that is, $\|W\,\mathbf{v}\|_1$ is $\sum_i |w_i||v_i|$. (If W is a general $m \times m$ nonsingular matrix then $\|W\,\mathbf{v}\|_1$ is called the *weighted 1-norm* of $\mathbf{v}$.) The norm or weighted norm of an integral vector $\mathbf{v} \in \mathbb{Z}^m$ is defined by considering $\mathbf{v}$ to be in $\mathbb{R}^m$. We now state in words the problem of optimal homologous chains and later formalize it in (2):

> Given a p-chain $\mathbf{c}$ in K and a diagonal matrix W of appropriate dimension, the optimal homologous chain problem (OHCP) is to find a chain $\mathbf{c}^*$ which has the minimal 1-norm $\|W\mathbf{c}^*\|_1$ among all chains homologous to $\mathbf{c}$.

Remark 3.1. In the natural case where simplices are weighted and the optimality of the chains is to be determined with respect to these weights, we may take W to be diagonal with w_i being the weight of simplex σ_i. In our formulation some of the weights can be 0. Notice that the signs of the simplex weights are ignored in our formulation since we only work with norms.

Remark 3.2. In Section 1 we surveyed the computational topology literature on the problem of finding optimal homologous *cycles*. The flexibility of our formulation allows us to solve the more general, optimal homologous *chain* problem, with the cycle case being a special case requiring no modification in the equations, algorithm, or theorems.

Remark 3.3. The choice of 1-norm is important. At first, it might seem easier to pose OHCP using 2-norm. Then, calculus can be used to pose the minimization as a stationary point problem when OHCP is formulated with only equality constraints which appear in (2) below. This case can be solved as a linear system of equations. By using 1-norm instead of 2-norm, we have to solve a linear program (as we will show below) instead of a linear system. But in return, we are able to get integer valued solutions when the appropriate conditions are satisfied.

The formulation of OHCP is the *weighted ℓ^1-optimization* of homologous chains. This is very general and allows for different types of optimality to be achieved by choosing different weight matrices. For example, assume that the simplicial complex K of dimension greater than p is embedded in $\mathbb{R}^d$, where $d \geq p + 1$. Let W be a diagonal matrix with the i-th diagonal entry being the Euclidean p-dimensional volume of a p-simplex. This specializes the problem to the *Euclidean ℓ^1-optimization* problem. The resulting optimal chain has the smallest p-dimensional volume amongst all chains homologous to the given one. If W is taken to be the identity matrix, with appropriate additional conditions to the above formulation, one can solve the *ℓ^0-optimization* problem. The resulting optimal solution has the smallest *number* of p-simplices amongst all chains homologous to $\mathbf{c}$, as we show in Section 3.2.

The central idea of this paper consists of the following steps: (i) write OHCP as an integer program involving 1-norm minimization, subject to linear constraints; (ii) convert the integer program into an integer *linear* program by converting the 1-norm cost function to a linear one using the standard technique of introducing some extra variables and constraints; (iii) find the conditions under which the constraint matrix of the integer linear program is totally unimodular; and (iv) for this class of problems, relax the integer linear program to a linear program by dropping the constraint that the variables be integral. The resulting optimal chain obtained by solving the linear program will be an integer valued chain homologous to the given chain.

3.1 Optimal homologous chains and linear programming

Now we formally pose OHCP as an optimization problem. After showing existence of solutions we reformulate the optimization problem as an integer linear program and eventually as a linear program.

Assume that the number of p- and $(p+1)$-simplices in K is m and n respectively, and let W be a diagonal $m \times m$ matrix. Given an integer valued p-chain $\mathbf{c}$ the optimal homologous chain problem is to solve:

$$\boxed{\min_{\mathbf{x},\mathbf{y}} \|W\,\mathbf{x}\|_1 \quad \text{such that} \quad \mathbf{x} = \mathbf{c} + [\partial_{p+1}]\,\mathbf{y},\ \text{and}\ \mathbf{x} \in \mathbb{Z}^m,\ \mathbf{y} \in \mathbb{Z}^n\,.} \tag{2}$$

In the problem formulation (2) we have given no indication of the algorithm that will be used to solve the problem. Before we develop the computational side, it is important to show that a solution to this problem always exists.

Claim 3.4. *For any given p-chain $\mathbf{c}$ and any matrix W, the solution to problem* (2) *exists.*

Proof. Define the set

$$U_{\mathbf{c}} := \{\|W\,\mathbf{x}\|_1 \ \mid\ \mathbf{x} = \mathbf{c} + [\partial_{p+1}]\,\mathbf{y},\ \mathbf{x} \in \mathbb{Z}^m \text{ and } \mathbf{y} \in \mathbb{Z}^n\}\,.$$

We show that this set has a minimum which is contained in the set. Consider the subset $U'_{\mathbf{c}} \subseteq U_{\mathbf{c}}$ defined by

$$U'_{\mathbf{c}} = \{\|W\,\mathbf{x}\|_1 \;|\; \|W\,\mathbf{x}\|_1 \leq \|W\,\mathbf{c}\|_1,\; \mathbf{x} = \mathbf{c} + [\partial_{p+1}]\,\mathbf{y},\; \mathbf{x} \in \mathbb{Z}^m \text{ and } \mathbf{y} \in \mathbb{Z}^n\}\,.$$

This set $U'_{\mathbf{c}}$ is finite since $\mathbf{x}$ is integral. Therefore, $\inf U_{\mathbf{c}} = \inf U'_{\mathbf{c}} = \min U'_{\mathbf{c}}$. □

In the rest of this paper we assume that W is a diagonal matrix obtained from *weights* on simplices as follows. Let w be a real-valued weight function on the oriented p-simplices of K and let W be the corresponding diagonal matrix (the i-th diagonal entry of W is $w(\sigma_i) = w_i$).

The resulting objective function $\|W\,\mathbf{x}\|_1 = \sum_i |w_i|\,|x_i|$ in (2) is not linear in x_i because it uses the absolute value of x_i. It is however, piecewise-linear in these variables. As a result, (2) can be reformulated as an integer *linear* program in the following standard way [1, page 18]:

$$\begin{aligned} &\min \sum_i |w_i|\,(x_i^+ + x_i^-) \\ &\text{subject to} \quad \mathbf{x}^+ - \mathbf{x}^- = \mathbf{c} + [\partial_{p+1}]\,\mathbf{y} \\ &\qquad \mathbf{x}^+,\; \mathbf{x}^- \geq \mathbf{0} \\ &\qquad \mathbf{x}^+,\, \mathbf{x}^- \in \mathbb{Z}^m,\; \mathbf{y} \in \mathbb{Z}^n\,. \end{aligned} \tag{3}$$

Comparing the above formulation to the standard form integer linear program in (1), note that the vector $\mathbf{x}$ in (1) corresponds to $[\mathbf{x}^+,\, \mathbf{x}^-,\, \mathbf{y}]^T$ in (3) above. Thus the minimization is over $\mathbf{x}^+$, $\mathbf{x}^-$ and $\mathbf{y}$, and the coefficients of x_i^+ and x_i^- in the objective function are $|w_i|$, but the coefficients corresponding to y_j are zero. The linear programming relaxation of this formulation just removes the constraints about the variables being integral. The resulting linear program is:

$$\begin{aligned} &\min \sum_i |w_i|\,(x_i^+ + x_i^-) \\ &\text{subject to} \quad \mathbf{x}^+ - \mathbf{x}^- = \mathbf{c} + [\partial_{p+1}]\,\mathbf{y} \\ &\qquad \mathbf{x}^+,\; \mathbf{x}^- \geq \mathbf{0}\,. \end{aligned} \tag{4}$$

To use the result about standard form polyhedron in Theorem 2.1 we can eliminate the free (unrestricted in sign) variables $\mathbf{y}$ by replacing these by $\mathbf{y}^+ - \mathbf{y}^-$ and imposing the non-negativity constraints on the new variables [1, page 5]. The resulting linear program has the same objective function, and the equality constraints:

$$\mathbf{x}^+ - \mathbf{x}^- = \mathbf{c} + [\partial_{p+1}]\,(\mathbf{y}^+ - \mathbf{y}^-)\,, \tag{5}$$

and thus the equality constraint matrix is $\begin{bmatrix} I & -I & -B & B \end{bmatrix}$, where $B = [\partial_{p+1}]$. We now prove a result about the total unimodularity of this matrix.

Lemma 3.5. *If $B = [\partial_{p+1}]$ is totally unimodular then so is the matrix $\begin{bmatrix} I & -I & -B & B \end{bmatrix}$.*

Proof. The proof uses operations that preserve the total unimodularity of a matrix. These are listed in [15, page 280]. If B is totally unimodular then so is the matrix $\begin{bmatrix} -B & B \end{bmatrix}$ since scalar multiples of columns of B are being appended on the left to get this matrix. The full matrix in question can be obtained from this one by appending columns with a single ± 1 on the left, which proves the result. □

As a result of Corollary 2.2 and Lemma 3.5, we have the following *algorithmic* result.

Theorem 3.6. *If the boundary matrix $[\partial_{p+1}]$ of a finite simplicial complex of dimension greater than p is totally unimodular, the optimal homologous chain problem* (2) *for p-chains can be solved in polynomial time.*

Proof. We have seen above that a reformulation of OHCP (2), without the integrality constraints, leads to the linear program (4). By Lemma 3.5, the equality constraint matrix of this linear program is totally unimodular. Then by Corollary 2.2 the linear program (4) can be solved in polynomial time, while achieving an integral solution. ☐

Remark 3.7. One may wonder why Theorem 3.6 does not work when $\mathbb{Z}_2$-valued chains are considered instead of integer-valued chains. We could simulate $\mathbb{Z}_2$ arithmetic while using integers or reals by modifying (2) as follows:

$$\min_{\mathbf{x},\,\mathbf{y}} \|W\,\mathbf{x}\|_1 \quad \text{such that} \quad \mathbf{x} + 2\,\mathbf{u} = \mathbf{c} + [\partial_{p+1}]\,\mathbf{y},\ \text{and}\ \mathbf{x} \in \{0,1\}^m,\ \mathbf{u} \in \mathbb{Z}^m,\ \mathbf{y} \in \mathbb{Z}^n\,. \tag{6}$$

The trouble is that the coefficient 2 of $\mathbf{u}$ destroys the total unimodularity of the constraint matrix in the linear programming relaxation of the above formulation, even when $[\partial_{p+1}]$ is totally unimodular. Thus we cannot solve the above integer program as a linear program and still get integer solutions.

Remark 3.8. We can associate weights with $(p+1)$-simplices while formulating the optimization problem (2). Then, we could minimize $\|W\mathbf{z}\|_1$ where $\mathbf{z} = [\mathbf{x}, \mathbf{y}]^T$. In that case, we obtain a p-chain c^* homologous to the given chain c and also a $(p+1)$-chain d whose boundary is $c^* - c$ and the weights of c^* and d together are the smallest. If the given cycle c is null homologous, the optimal y would be an optimal $(p+1)$-chain bounded by c.

Remark 3.9. The simplex method and its variants search only the basic feasible solutions (vertices of the constraint polyhedron), while choosing ones that never make the objective function worse. Thus if the polyhedron is integral, one could stop the simplex method at any step before reaching optimality and still obtain an integer valued homologous chain whose norm is no worse than that of the given chain.

3.2 Minimizing the number of simplices

The general weighted ℓ^1-optimization problem (2) can be specialized by choosing different weight matrices. One can also solve variations of the OHCP problem by adding other constraints which do not destroy the total unimodularity of the constraint matrix. We consider one such specialization here – that of finding a homologous chain with the smallest *number* of simplices.

If the matrix W is chosen to be the identity matrix, then one can solve the ℓ^0-optimization problem by solving a modified version of the ℓ^1-optimization problem (2). One just imposes the extra condition that every entry of $\mathbf{c}$ and $\mathbf{x}$ be in $\{-1, 0, 1\}$. With this choice of $W = I$ and with $\mathbf{c} \in \{-1, 0, 1\}^m$, the problem (2) becomes:

$$\min_{\mathbf{x},\,\mathbf{y}} \|\mathbf{x}\|_1 \quad \text{such that} \quad \mathbf{x} = \mathbf{c} + [\partial_{p+1}]\,\mathbf{y},\ \text{and}\ \mathbf{x} \in \{-1,0,1\}^m,\ \mathbf{y} \in \mathbb{Z}^n\,. \tag{7}$$

Theorem 3.10. *For any given p-chain $\mathbf{c} \in \{-1, 0, 1\}^m$, a solution to problem* (7) *exists. Furthermore, amongst all $\mathbf{x}$ homologous to $\mathbf{c}$, the optimal homologous chain $\mathbf{x}^*$ has the smallest number of nonzero entries, that is, it is the ℓ^0-optimal homologous chain.*

Proof. The proof of existence is identical to the proof of Claim 3.4. The condition that $\mathbf{c}$ takes values in -1, 0, 1 ensures that at least $\mathbf{x} = \mathbf{c}$ can be taken as the solution if no other homologous chain exists. For the ℓ^0-optimality, note that since the entries of the optimal solution $\mathbf{x}^*$ are constrained to be in $\{-1, 0, 1\}$, the 1-norm measures the number of nonzero entries. Thus the 1-norm optimal solution is also the one with the smallest number of non-zero entries. ☐

Remark 3.11. Note that even with the given chain $\mathbf{c}$ taking values in $\{-1, 0, 1\}$, without the extra constraint that $\mathbf{x} \in \{-1, 0, 1\}^m$ (rather than just $\mathbf{x} \in \mathbb{Z}^m$), the optimal 1-norm solution components may take values

outside $\{-1, 0, 1\}$. For example, consider the simplicial complex K triangulating a cylinder which is shaped like an hourglass. Let c_1 and c_2 be the two boundary cycles of the hour glass so that $c_1 + c_2$ is not trivial. Let z be the smallest cycle around the middle of the hour glass which is homologous to each of c_1 and c_2. Since $c_1 + c_2 = 2z$, the optimal cycle homologous to $c_1 + c_2$ has values 2 or -2 for some edges even if c_1 and c_2 have values only in $\{-1, 0, 1\}$ for all edges. It may or may not be true that the number of nonzero entries is minimal in such an optimal solution. We have not proved it either way. But Theorem 3.10 provides a guarantee for computing ℓ^0-optimal solution when the additional constraints are placed on $\mathbf{x}$.

The linear programming relaxation of problem (7) is

$$\begin{aligned} \min \; & \sum_i (x_i^+ + x_i^-) \\ \text{subject to} \;\; & \mathbf{x}^+ - \mathbf{x}^- = \mathbf{c} + [\partial_{p+1}]\, \mathbf{y} \\ & \mathbf{x}^+,\, \mathbf{x}^- \leq \mathbf{1} \\ & \mathbf{x}^+,\, \mathbf{x}^- \geq \mathbf{0}\,. \end{aligned} \tag{8}$$

One can show the integrality of the feasible set polyhedron by using slack variables to convert the inequalities $\mathbf{x}^+ \leq \mathbf{1}$ and $\mathbf{x}^- \leq \mathbf{1}$ to equalities and then using the $\mathcal{P}$ form of the polyhedron from Theorem 2.1. Equivalently, all the constraints can be written as inequalities and the $\mathcal{Q}$ polyhedron can be used. For a change we choose the latter method here. Writing the constraints as inequalities, in matrix form the constraints are

$$\begin{bmatrix} -I & I & B & -B \\ I & -I & -B & B \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{y}^+ \\ \mathbf{y}^- \end{bmatrix} \geq \begin{bmatrix} -\mathbf{c} \\ \mathbf{c} \\ -\mathbf{1} \\ -\mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{9}$$

where $B = [\partial_{p+1}]$. Then analogously to Lemma 3.5 and Theorem 3.6 the following are true.

Lemma 3.12. *If $B = [\partial_{p+1}]$ is totally unimodular then so is the constraint matrix in* (9).

Theorem 3.13. *If the boundary matrix $[\partial_{p+1}]$ of a finite simplicial complex of dimension greater than p is totally unimodular, then given a p-chain that takes values in $\{-1, 0, 1\}$, a homologous p-chain with the smallest number of non-zeros taking values in $\{-1, 0, 1\}$ can be found in polynomial time.*

In subsequent sections, we characterize the simplicial complexes for which the boundary matrix $[\partial_{p+1}]$ is totally unimodular. These are the main theoretical results of this paper, formalized as Theorems 4.1, 5.2, and 5.7.

4 Manifolds

Our results in Section 5.1 are valid for *any* finite simplicial complex. But first we consider a simpler case – simplicial complexes that are triangulations of manifolds. We show that for finite triangulations of compact p-dimensional *orientable* manifolds, the top non-trivial boundary matrix $[\partial_p]$ is totally unimodular irrespective of the orientations of its simplices. We also give examples of non-orientable manifolds where total unimodularity does not hold. Further examination of why total unimodularity does not hold in these cases leads to our main results in Theorems 5.2.

4.1 Orientable manifolds

Let K be a finite simplicial complex that triangulates a $(p+1)$-dimensional compact orientable manifold M. As before, let $[\partial_{p+1}]$ be the matrix corresponding to $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$ in the elementary chain bases.

Theorem 4.1. *For a finite simplicial complex triangulating a $(p+1)$-dimensional compact orientable manifold, $[\partial_{p+1}]$ is totally unimodular irrespective of the orientations of the simplices.*

Proof. First, we prove the theorem assuming that the $(p+1)$-dimensional simplices of K are oriented *consistently*. Then, we argue that the result still holds when orientations are arbitrary.

Consistent orientation of $(p+1)$-simplices means that they are oriented in such a way that for the $(p+1)$-chain c, which takes the value 1 on each oriented $(p+1)$-simplex in K, $\partial_{p+1} c$ is carried by the topological boundary ∂M of M. If M has no boundary then $\partial_{p+1} c$ is 0. It is known that consistent orientation of $(p+1)$-simplices always exists for a finite triangulation of a compact orientable manifold. Therefore, assume that the given triangulation has consistent orientation for the $(p+1)$-simplices. The orientation of the p- and lower dimensional simplices can be chosen arbitrarily.

Each p-face τ is the face of either one or two $(p+1)$-simplices (depending on whether τ is a boundary face or not). Thus the row of $[\partial_{p+1}]$ corresponding to τ contains one or two nonzeros. Such a nonzero entry is 1 if the orientation of τ agrees with that of the corresponding $(p+1)$-simplex and -1 if it does not.

Heller and Tompkins [13] gave a sufficient condition for the unimodularity of $\{-1, 0, 1\}$-matrices whose columns have no more than two nonzero entries. Such a matrix is totally unimodular if its rows can be divided into two partitions (one possibly empty) with the following condition. If two nonzeros in a column belong to the same partition, they must be of opposite signs, otherwise they must be in different row partitions. Consider $[\partial_{p+1}]^T$, the transpose of $[\partial_{p+1}]$. Each column of $[\partial_{p+1}]^T$ contains at most two nonzero entries, and if there are two then they are of opposite signs because of the consistent orientations of the $(p+1)$-dimensional simplices. In this case, the simple division of rows into two partitions with one containing all rows and the other empty works. Thus $[\partial_{p+1}]^T$ and hence $[\partial_{p+1}]$ is totally unimodular.

Now, reversing the orientation of a $(p+1)$-simplex means that the corresponding column of $[\partial_{p+1}]$ be multiplied by -1. This column operation preserves the total unimodularity of $[\partial_{p+1}]$. Since any arbitrary orientation of the $(p+1)$-simplices can be obtained by preserving or reversing their orientations in a consistent orientation, we have the result as claimed. □

As a result of the above theorem and Theorem 3.6 we have the following result.

Corollary 4.2. *For a finite simplicial complex triangulating a $(p+1)$-dimensional compact orientable manifold, the optimal homologous chain problem can be solved for p-dimensional chains in polynomial time.*

The result in Corollary 4.2 when specialized to $\mathbb{R}^{p+1}$ also appears in [19] though the reasoning is different.

4.2 Non-orientable manifolds

For non-orientable manifolds we give two examples which show that total unimodularity may not hold in this case. We also discuss the role of torsion in these examples in preparation for Theorem 5.2.

Our first example is the Möbius strip and the second one is the projective plane. Simplicial complexes for these two non-orientable surfaces are shown in Figure 1. The boundary matrices $[\partial_2]$ for these simplicial complexes are given in the Appendix in (11) and (12).

Let M be the Möbius strip. We consider its absolute homology $H_1(M)$ and its relative homology $H_1(M, \partial M)$ relative to its boundary. Consult [14, page 135] to see how the various homology groups are

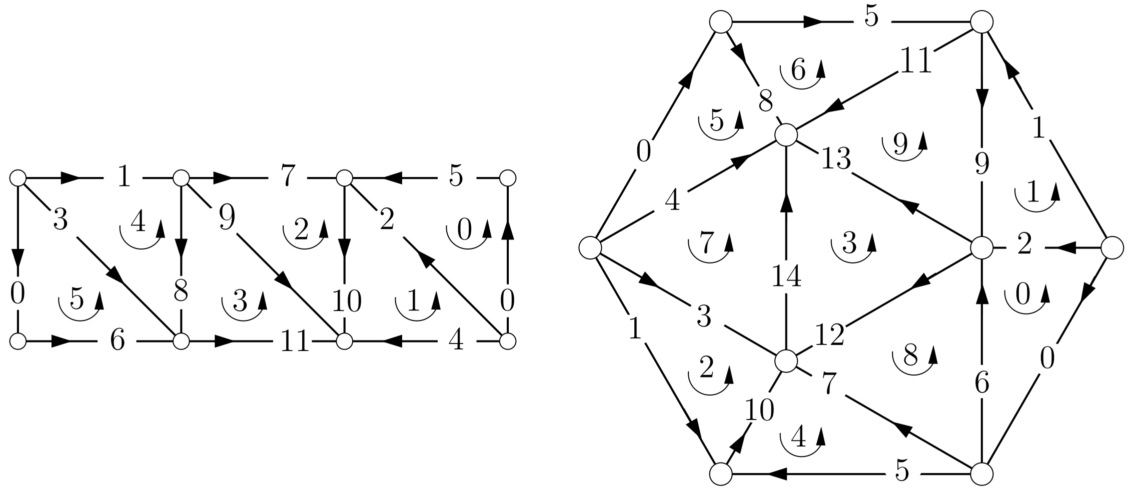

Figure 1: Triangulations of two non-orientable manifolds, shown as abstract simplicial complexes. The left figure shows a triangulation of the Möbius strip and the right one shows the projective plane. The numbers are the edge and triangles numbers. These correspond to the row and column numbers of the matrices (11) and (12).

calculated using an exact sequence. We note that $H_1(M) \cong \mathbb{Z}$, that is, its H_1 group has no torsion. This can be seen by reducing the matrix (11) in the Appendix to Smith normal form (SNF). The SNF for the matrix consists of a 6×6 identity matrix on the top and a zero block below, which implies the absence of torsion.

Let K be the simplicial complex triangulating M. Consider a submatrix S of the matrix $[\partial_2]$ shown in Appendix as (11). This submatrix is formed by selecting the columns in the order 5, 4, 3, 2, 1, 0. From the matrix thus formed, select the rows 0, 3, 8, 9, 10, 2 in that order. This selection of rows and columns corresponds to all the triangles and the edges encountered as one goes from left to right in the Möbius triangulation shown in Figure 1. The resulting submatrix is

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The determinant of this matrix is -2 and this shows that the boundary matrix is not totally unimodular. The SNF for this matrix, it turns out, *does* reveal the torsion. This matrix S is the relative boundary matrix $\partial_2^{(L,L_0)}$ where $L = K$ and L_0 are the edges in $\partial\, M$. The SNF has 1's along the diagonal and finally a 2. This is an example where there is no torsion in the absolute homology but some torsion in the relative homology and the boundary matrix is not totally unimodular. We formulate this condition precisely in Theorem 5.2.

The matrix $[\partial_2]$ given in Appendix as (12) for the projective plane triangulation is much larger. But it is easy to find a submatrix with determinant greater than 1. This can be done by finding the Möbius strip in the triangulation of the projective plane. For example if one traverses from top to bottom in the triangulation of the projective plane in Figure 1 the triangles encountered correspond to columns 6, 9, 3, 8, 4 of (12) and the

edges correspond to rows 5, 11, 13, 12, 7. The corresponding submatrix is

$$S = \begin{bmatrix} -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

and its determinant is -2. Thus the boundary matrix (12) is not totally unimodular. Again, we observe that there is relative torsion in $H_1(L, L_0)$ for the subcomplexes corresponding to the selection of S from $[\partial_2]$. Here L consists of the triangles specified above, which form a Möbius strip in the projective plane. The subcomplex L_0 consists of the edges forming the boundary of this strip. This connection between submatrices and relative homology is examined in the next section.

5 Simplicial complexes

Now we consider the more general case of simplicial complexes. Our result in Theorem 5.2 characterizes the total unimodularity of boundary matrices for arbitrary simplicial complexes. Since we do not use any conditions about the geometric realization or embedding in $\mathbb{R}^n$ for the complex, the result is also valid for abstract simplicial complexes. As a corollary of the characterization we show that the OHCP can be solved in polynomial time as long as the input complex satisfies a torsion-related condition.

5.1 Total unimodularity and relative torsion

Let K be a finite simplicial complex of dimension greater than p. We will need to refer to its subcomplexes formed by the union of some of its simplices of a specific dimension. This is formalized in the definition below.

Definition 5.1. A *pure simplicial complex* of dimension p is a simplicial complex formed by a collection of p-simplices and their proper faces. Similarly, a *pure subcomplex* is a subcomplex that is a pure simplicial complex.

An example of a pure simplicial complex of dimension p is one that triangulates a p-dimensional manifold. Another example, relevant to our discussion, is a subcomplex formed by a collection of some p-simplices of a simplicial complex and their lower dimensional faces.

Let $L \subseteq K$ be a pure subcomplex of dimension $p+1$ and $L_0 \subset L$ be a pure subcomplex of dimension p. If $[\partial_{p+1}]$ is the matrix representing $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$, then the matrix representing the relative boundary operator

$$\partial_{p+1}^{(L,L_0)} : C_{p+1}(L, L_0) \to C_p(L, L_0)\,,$$

is obtained by first *including* the columns of $[\partial_{p+1}]$ corresponding to $(p+1)$-simplices in L and then, from the submatrix so obtained, *excluding* the rows corresponding to the p-simplices in L_0 and any zero rows. The zero rows correspond to p-simplices that are not faces of any of the $(p+1)$-simplices of L.

As before, let $[\partial_{p+1}]$ be the matrix of ∂_{p+1} in the elementary chain bases for K. Then the following holds.

Theorem 5.2. *$[\partial_{p+1}]$ is totally unimodular if and only if $H_p(L, L_0)$ is torsion-free, for all pure subcomplexes L_0, L of K of dimensions p and $p+1$ respectively, where $L_0 \subset L$.*

Proof. ($\Rightarrow$) We show that if $H_p(L, L_0)$ has torsion for some L, L_0 then $[\partial_{p+1}]$ is not totally unimodular. Let $\left[\partial_{p+1}^{(L,L_0)}\right]$ be the corresponding relative boundary matrix. Bring $\left[\partial_{p+1}^{(L,L_0)}\right]$ to Smith normal form using the reduction algorithm [14][pages 55–57]. This is a block matrix

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where $D = \mathrm{diag}(d_1, \ldots, d_l)$ is a diagonal matrix and the block row or column of zero matrices shown above may be empty, depending on the dimension of the matrix. Recall that d_i are integers and $d_i \geq 1$. Moreover, since $H_p(L, L_0)$ has torsion, $d_k > 1$ for some $1 \leq k \leq l$. Thus the product $d_1 \ldots d_k$ is greater than 1. By a result of Smith [17] quoted in [15, page 50], this product is the greatest common divisor of the determinants of all $k \times k$ square submatrices of $\left[\partial_{p+1}^{(L,L_0)}\right]$. But this implies that some square submatrix of $\left[\partial_{p+1}^{(L,L_0)}\right]$, and hence of $[\partial_{p+1}]$, has determinant magnitude greater than 1. Thus $[\partial_{p+1}]$ is not totally unimodular.

($\Leftarrow$) Assume that $[\partial_{p+1}]$ is not totally unimodular. We will show that then there exist subcomplexes L_0 and L of dimensions p and $(p+1)$ respectively, with $L_0 \subset L$, such that $H_p(L, L_0)$ has torsion. Let S be a square submatrix of $[\partial_{p+1}]$ such that $|\det(S)| > 1$. Let L correspond to the columns of $[\partial_{p+1}]$ that are *included* in S and let B_L be the submatrix of $[\partial_{p+1}]$ formed by these columns. This submatrix B_L may contain zero rows. Those zero rows (if any) correspond to p-simplices that do not occur as a face of any of the $(p+1)$-simplices in L. In order to form S from B_L, these zero rows can first be safely discarded to form a submatrix B'_L. This is because $\det(S) \neq 0$ and so these zero rows cannot occur in S.

The rows in B'_L correspond to p-simplices that occur as a face of some $(p+1)$-simplex in L. Let L_0 correspond to rows of B'_L which are *excluded* to form S. Now S is the matrix representation of the relative boundary matrix $\partial_p^{(L,L_0)}$. Reduce S to Smith normal form. The normal form is a square diagonal matrix. Since the elementary row and column operations preserve determinant magnitude, the determinant of the resulting diagonal matrix has magnitude greater than 1. Thus at least one of the diagonal entries in the normal form is greater than 1. But then by [14, page 61] $H_p(L, L_0)$ has torsion. □

Remark 5.3. The characterization appears to be no easier to check than the definition of total unimodularity since it involves checking *every* L, L_0 pair. However, it is also no *harder* to check than total unimodularity. This leads to the following result of possible interest in computational topology and matroid theory.

Corollary 5.4. *For a simplicial complex K of dimension greater than p, there is a polynomial time algorithm for answering the following question: Is $H_p(L, L_0)$ torsion-free for* all *subcomplexes L_0 and L of dimensions p and $(p+1)$ such that $L_0 \subset L$?*

Proof. Seymour's decomposition theorem for totally unimodular matrices [16],[15, Theorem 19.6] yields a polynomial time algorithm for deciding if a matrix is totally unimodular or not [15, Theorem 20.3]. That algorithm applied on the boundary matrix $[\partial_{p+1}]$ proves the above assertion. □

Remark 5.5. Note that the naive algorithm for the above problem is clearly exponential. For every pair L, L_0 one can use a polynomial time algorithm to find the Smith normal form. But the number of L, L_0 pairs is exponential in the number of p and $(p+1)$-simplices of K.

Remark 5.6. The same polynomial time algorithm answers the question : Does $H_p(L, L_0)$ have torsion for *some* pair L, L_0 ?

5.2 A special case

In Section 4 we have seen the special case of compact orientable manifolds. We saw that the top dimensional boundary matrix of a finite triangulation of such a manifold is totally unimodular. Now we show another special case for which the boundary matrix is totally unimodular and hence OHCP is polynomial time solvable. This case occurs when we ask for optimal d-chains in a simplicial complex K which is embedded in $\mathbb{R}^{d+1}$. In particular, OHCP can be solved by linear programming for 2-chains in 3-complexes embedded in $\mathbb{R}^3$. This follows from the following result:

Theorem 5.7. *Let K be a finite simplicial complex embedded in $\mathbb{R}^{d+1}$. Then, $H_d(L, L_0)$ is torsion-free for all pure subcomplexes L_0 and L of dimensions d and $d+1$ respectively, such that $L_0 \subset L$.*

Proof. We consider the $(d+1)$-dimensional relative *cohomology group* $H^{d+1}(L, L_0)$ (See [14] for example). It follows from the Universal Coefficient Theorem for cohomology [14, Theorem 53.1] that

$$H^{d+1}(L, L_0) = \mathrm{Hom}(H_{d+1}(L, L_0), \mathbb{Z}) \oplus \mathrm{Ext}(H_d(L, L_0), \mathbb{Z})$$

where Hom is the group of all homomorphisms from $H_{d+1}(L, L_0)$ to $\mathbb{Z}$ and Ext is the group of all of extensions between $H_d(L, L_0)$ and $\mathbb{Z}$. These definitions can be found in [14, Chapter 5 and 7]. The main observation is that if $H_d(L, L_0)$ has torsion, $\mathrm{Ext}(H_d(L, L_0), \mathbb{Z})$ has torsion and hence $H^{d+1}(L, L_0)$ has torsion.

On the other hand, by Alexander Spanier duality [18, page 296]

$$H^{d+1}(L, L_0) = H_0(\mathbb{R}^{d+1} \setminus |L_0|, \mathbb{R}^{d+1} \setminus |L|)$$

where $|L|$ denotes the underlying space of L. Since 0-dimensional homology groups cannot have torsion, $H^{d+1}(L, L_0)$ cannot have torsion. We reach a contradiction. $\square$

Corollary 5.8. *Given a d-chain c in a weighted finite simplicial complex embedded in $\mathbb{R}^{d+1}$, an optimal chain homologous to c can be computed by a linear program.*

Proof. Follows from Theorem 5.7, Theorem 5.2, and Theorem 2.2. $\square$

5.3 Total unimodularity and Möbius complexes

As another special case, we provide a characterization of the total unimodularity of $(p+1)$-boundary matrix of simplicial complexes in terms of a forbidden complex called Möbius complex, for $p \leq 1$. In contrast to the previous characterization (in terms of relative homology of K), we directly employ certain results on totally unimodular matrices to derive this characterization in terms of submatrices called cycle matrices. We show in Theorem 5.13 that the $(p+1)$-boundary matrix of a finite simplicial complex for $p \leq 1$ is totally unimodular if and only if the input complex does not have a $(p+1)$-dimensional Möbius complex as a subcomplex. In particular, this observation along with Theorem 5.2 implies that a 2-complex does not have relative torsion if and only if it does not have a Möbius complex in it. We also demonstrate by example that this result does not generalize to higher values of p.

Definition 5.9. A $(p+1)$-dimensional *cycle complex* is a sequence $\sigma_0, \ldots, \sigma_{k-1}$ of $(p+1)$-simplices such that σ_i and σ_j have a common face if and only if $j = (i+1) \pmod{k}$ and that common face is a p-simplex. Such a cycle complex triangulates a $(p+1)$-manifold. We call it a $(p+1)$-dimensional *cylinder complex* if it is orientable and a $(p+1)$-dimensional *Möbius complex* if it is nonorientable.

Definition 5.10. For $k \geq 2$, a $k \times k$ matrix C is called a *k-cycle matrix (k-CM)* if $C_{ij} \in \{-1, 0, 1\}$, and C has the following form up to row and column permutations and scalings by -1:

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \beta \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}, \ \beta = \pm 1. \tag{10}$$

A k-CM with $\beta = (-1)^k$ is termed a *cylinder cycle matrix* (k-CCM), while one with $\beta = (-1)^{k+1}$ is termed a *Möbius cycle matrix* (k-MCM). We will refer to the form shown in (10) as the *normal form* cycle matrix.

As an example, consider a triangulation K of a Möbius strip with $k \geq 5$ triangles shown in Figure 2. Let K_0 be the complex for the boundary of the Möbius strip. In the figure, K_0 consists of the horizontal edges. Then the relative boundary matrix $\left[\partial_2^{(K,K_0)}\right]$ of the Möbius strip K modulo its edge K_0 is a k-MCM. The orientations of triangle τ_{k-1} and that of the terminal edge e_0 are opposite if k is even, but the orientations agree if k is odd, giving $\beta = (-1)^{k+1}$. Note that in Section 4.2, the submatrix S of the boundary matrix of the Möbius strip was such a relative boundary matrix and it is an example of a 6-MCM. Another example in that section was the 5-MCM obtained from the boundary matrix of the projective plane.

Similarly, we observe a k-CCM as the relative boundary-2 matrix of a cylinder triangulated with k triangles, modulo the cylinder's edges. Reversing the orientation of an edge or a triangle results in scaling the corresponding row or column, respectively, of the boundary matrix by -1. These examples motivate the names "Möbius" and "cylinder" matrices – a cycle matrix can be interpreted as the relative boundary matrix of a Möbius or cylinder complex. So, we have the following result.

Lemma 5.11. *Let K be a finite simplicial complex of dimension greater than p. The boundary matrix $[\partial_{p+1}]$ has no k-*MCM *for any $k \geq 2$ if and only if K does not have any $(p+1)$-dimensional Möbius complex as a subcomplex.*

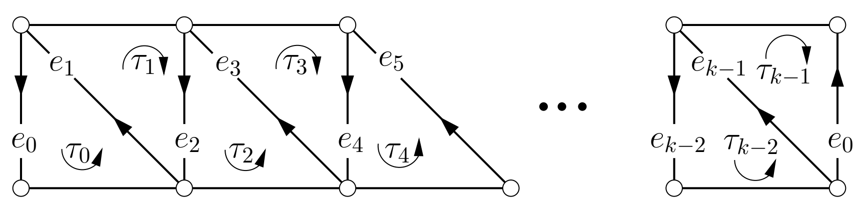

Figure 2: Triangulation of a Möbius strip with k triangles.

It is now easy to see that the absence of Möbius complexes is a necessary condition for total unimodularity. We show that this condition is also sufficient for 2- or lower dimensional complexes. We first need the simple result that an MCM is not totally unimodular.

Lemma 5.12. *Let C be a k-CM. Then* $\det C = 0$ *if it is a k-CCM, and* $|\det C| = 2$ *if it is a k-MCM.*

Proof. The matrix C can always be brought into the normal form with a series of row and column exchanges and scalings by -1. Note that these operations preserve the value of $|\det C|$. Now assume that C has been brought into the normal form and call that matrix C'. We expand along the first row of C' to get $\det C' = 1 + (-1)^{k+1}\beta$, and the claim follows. $\square$

Theorem 5.13. *For $p \leq 1$, $[\partial_{p+1}]$ is totally unimodular if and only if the simplicial complex K has no Möbius subcomplex of dimension $p+1$.*

Proof. ($\Rightarrow$) If there is a Möbius subcomplex of dimension $p+1$ in K, then by Lemma 5.11 an MCM appears as a submatrix of $[\partial_{p+1}]$. That MCM is a certificate for $[\partial_{p+1}]$ not being totally unimodular since its determinant has magnitude 2 by Lemma 5.12.

($\Leftarrow$) Let K have no Möbius subcomplexes of dimension $p+1$. Then by Lemma 5.11, there are no MCMs as submatrices of $[\partial_{p+1}]$. Truemper [23, Theorem 28.3] has characterized all minimally non-totally unimodular matrices, i.e., matrices that are not totally unimodular, but their every proper submatrix is totally unimodular. These matrices belong to two classes, which Truemper denotes as $\mathscr{W}_1$ and $\mathscr{W}_7$. MCMs constitute the first class $\mathscr{W}_1$. A minimally non-totally unimodular matrix W is in $\mathscr{W}_7$ if and only if W has a row and a column containing at least four nonzeros each [23, Cor. 28.5]. Since $p \leq 1$, no column of $[\partial_{p+1}]$ can have four or more nonzeros, and hence no matrix from the class $\mathscr{W}_7$ can appear as a submatrix. Hence $[\partial_{p+1}]$ is totally unimodular if K has no $(p+1)$-dimensional Möbius subcomplexes. $\square$

The necessary condition in Theorem 5.13 extends beyond 2-complexes as Remark 5.14 indicates. However, we cannot extend the sufficiency condition; Remark 5.15 presents a counterexample.

Remark 5.14. Note that the absence of Möbius subcomplexes is a necessary condition for $[\partial_{p+1}]$ to be totally unimodular for *all* p. More precisely, if the simplicial complex K of dimension greater than p has a Möbius subcomplex of dimension $p+1$ then $[\partial_{p+1}]$ is not totally unimodular. By Lemma 5.11, an MCM appears as a submatrix of $[\partial_{p+1}]$ in this case. Its determinant has magnitude 2 by Lemma 5.12, trivially certifying that $[\partial_{p+1}]$ is not totally unimodular.

Remark 5.15. The characterization in Theorem 5.13 does not hold for higher values of p. We present a 3-complex which does not have a 3-dimensional Möbius subcomplex, but whose $[\partial_3]$ is not totally unimodular. Consider the simplicial complex consisting of the following seven tetrahedra formed from seven points numbered 0–6: $(0,1,2,3)$, $(0,1,2,4)$, $(0,1,2,5)$, $(0,1,2,6)$, $(0,1,3,4)$, $(0,2,3,5)$, $(1,2,3,6)$. It can be verified that the 19×7 boundary matrix $[\partial_3]$ of this simplicial complex has the 7×7 matrix

$$W = \begin{bmatrix} -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

as a submatrix where $\det(W) = -2$, certifying that $[\partial_3]$ is not totally unimodular. In fact, W is the *only* submatrix of $[\partial_3]$ which is not totally unimodular, and it belongs to the class $\mathscr{W}_7$ of minimally non-totally unimodular matrices.

6 Experimental Results

We have implemented our linear programming method to solve the optimal homologous chain problem. In Figure 3 we show some results of preliminary experiments.

The top row in Figure 3 shows the computation of optimal homologous 1-chains on the simplicial complex representation of a torus. The longer chain in each torus figure is the initial chain and the tighter shorter chain is the optimal homologous chain computed by our algorithm. The bottom row shows the result of the computation of an optimal 2-chain on a simplicial complex of dimension 3. The complex is the tetrahedral

triangulation of a solid annulus – a solid ball from which a smaller ball has been removed. Two cut-away views are shown. The outer surface of the sphere is the initial chain and the inner surface is computed as the optimal 2-chain.

In these experiments we used the linear program (4). The initial chains used had values in $\{-1, 0, 1\}$ on the simplices. In the torus examples for instance, the initial chain was 0 everywhere except along the initial curve shown. The curve was given an arbitrary orientation and the values of the chain on the edges forming the curve were $+1$ or -1 depending on the edge orientation. In these examples, the resulting optimal chains were oriented curves, with values of ± 1 on the edges along the curve. This is by no means guaranteed theoretically, as seen in the hour glass example in Remark 3.11. The only guarantee is that of integrality. However if it is essential that the optimal chain has values only in $\{-1, 0, 1\}$ then the linear program (8) or it's Euclidean variant can be used, imposing the additional constraint on the values of the optimal solution $\mathbf{x}$ as shown in linear program (8).

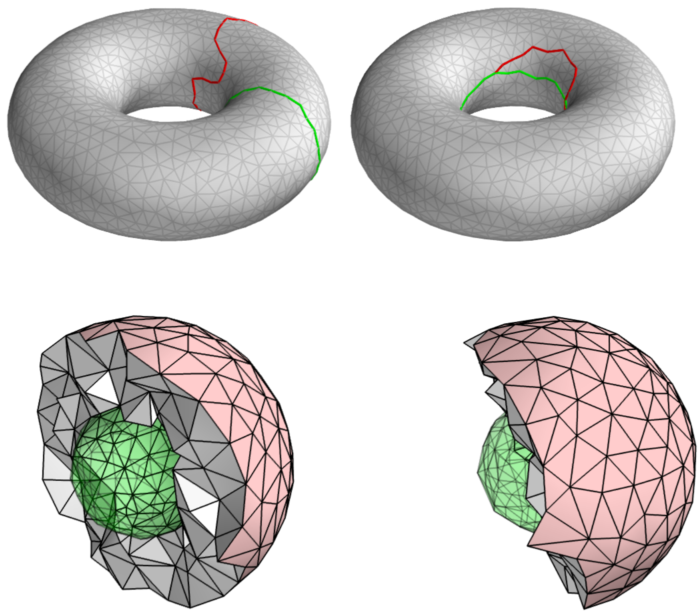

Figure 3: Some experimental results. See text for details on what is being computed here.

7 Discussion

Several questions crop up from our problem formulation and results. Instead of 1-norm $\|W\,\mathbf{x}\|_1$, we can consider minimizing $\sum_i w_i\, x_i$. In this case, the weights appear with signs and solutions may be unbounded. Nevertheless, our result in Theorem 3.6 remains valid. Of course, in this case we do not need to introduce x_i^+ and x_i^- since the objective function uses x_i rather than $|x_i|$. We may introduce more generalization in the OHCP formulation by considering a general matrix W instead of requiring it to be diagonal and then asking for minimizing $\|W\,\mathbf{x}\|_1$. We do not know if the corresponding optimization problem can be solved by a linear program. Can this optimization problem be solved in polynomial time for some interesting classes of complexes?

We showed that OHCP under $\mathbb{Z}$ coefficients can be solved by linear programs for a large class of topological spaces that have no relative torsion. This leaves a question for the cases when there is relative torsion. Is the problem NP-hard under such constraint? Taking the cue from our results, one can also ask

the following question. Even though we know that the problem is NP-hard under $\mathbb{Z}_2$ coefficients, is it true that OHCP in this case is polynomial time solvable at least for simplicial complexes that have no relative torsions (considered under $\mathbb{Z}$)? The answer is negative since OHCP for surfaces in $\mathbb{R}^3$ is NP-hard under $\mathbb{Z}_2$ coefficients [3] even though they are known to be torsion-free.

Even if the input complex has relative torsion, the constraint polyhedron of the linear program may still have vertices with integer coordinates. In that case, the linear program may still give an integer solution for chains that steer the optimization path toward such a vertex. In fact, we have observed experimentally that, for some 2-complexes with relative torsion, the linear program finds the integer solution for some input chains. It would be nice to characterize the class of chains for which the linear program still provides a solution even if the input complex has relative torsion.

A related question that has also been investigated recently is the problem of computing an optimal homology basis from a given complex. Again, positive results have been found for low dimensional cases such as surfaces [11] and one dimensional homology for simplicial complexes [5, 9]. The result of Chen and Freedman [4] implies that even this problem is NP-hard for high dimensional cycles under $\mathbb{Z}_2$. What about $\mathbb{Z}$? As in OHCP, would we have any luck here?

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References

[1] BERTSIMAS, D., AND TSITSIKLIS, J. N. *Introduction to Linear Optimization*. Athena Scientific, Belmont, MA., 1997.

[2] CHAMBERS, E. W., COLIN DE VERDIÈRE É., ERICKSON, J., LAZARUS, F., AND WHITTLESEY, K. Splitting (complicated) surfaces is hard. *Comput. Geom. Theory Appl. 41* (2008), 94–110.

[3] CHAMBERS, E. W., ERICKSON, J., AND NAYYERI, A. Minimum cuts and shortest homologous cycles. In *SCG '09: Proc. 25th Ann. Sympos. Comput. Geom.* (2009), pp. 377–385.

[4] CHEN, C., AND FREEDMAN, D. Hardness results for homology localization. In *SODA '10: Proc. 21st Ann. ACM-SIAM Sympos. Discrete Algorithms* (2010), pp. 1594–1604.

[5] CHEN, C., AND FREEDMAN, D. Measuring and computing natural generators for homology groups. *Computational Geometry 43*, 2 (2010), 169–181. Special Issue on the 24th European Workshop on Computational Geometry (EuroCG'08).

[6] COLIN DE VERDIÈRE É., AND ERICKSON, J. Tightening non-simple paths and cycles on surfaces. In *SODA '06: Proc. 17th Ann. ACM-SIAM Sympos. Discrete Algorithms* (2006), pp. 192–201.

[7] DE SILVA, V., AND GHRIST, R. Homological sensor networks. *Notices of the American Mathematical Society 54*, 1 (2007), 10–17.

[8] DEY, T. K., LI, K., SUN, J., AND COHEN-STEINER, D. Computing geometry-aware handle and tunnel loops in 3d models. In *SIGGRAPH '08: ACM SIGGRAPH 2008 papers* (New York, NY, USA, 2008), pp. 1–9.

[9] DEY, T. K., SUN, J., AND WANG, Y. Approximating loops in a shortest homology basis from point data. In *SCG '10: Proc. 26th Ann. Sympos. Comput. Geom.* (2010), pp. 166–175.

[10] EDELSBRUNNER, H., LETSCHER, D., AND ZOMORODIAN, A. Topological persistence and simplification. *Discrete Comput. Geom. 28* (2002), 511–533.

[11] ERICKSON, J., AND WHITTLESEY, K. Greedy optimal homotopy and homology generators. In *SODA '05: Proc. 16th Ann. ACM-SIAM Sympos. Discrete Algorithms* (2005), pp. 1038–1046.

[12] GÜLER, O., DEN HERTOG, D., ROOS, C., TERLAKY, T., AND TSUCHIYA, T. Degeneracy in interior point methods for linear programming: a survey. *Annals of Operations Research 46-47*, 1 (March 1993), 107–138.

[13] HELLER, I., AND TOMPKINS, C. B. An extension of a theorem of Dantzig's. In *Linear Inequalities and Related Systems*, H. W. Kuhn and A. W. Tucker, Eds., Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956, pp. 247–254.

[14] MUNKRES, J. R. *Elements of Algebraic Topology*. Addison–Wesley Publishing Company, Menlo Park, 1984.

[15] SCHRIJVER, A. *Theory of Linear and Integer Programming*. Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons Ltd., Chichester, 1986. A Wiley-Interscience Publication.

[16] SEYMOUR, P. D. Decomposition of regular matroids. *J. Combin. Theory Ser. B 28*, 3 (1980), 305–359.

[17] SMITH, H. J. S. On systems of linear indeterminate equations and congruences. *Philosophical Transactions of the Royal Society of London 151* (1861), 293–326.

[18] SPANIER, E. H. *Algebraic Topology*. McGraw-Hill Book Co., New York, 1966.

[19] SULLIVAN, J. M. *A Crystalline Approximation Theorem for Hypersurfaces*. PhD thesis, Princeton University, 1990.

[20] TAHBAZ-SALEHI, A., AND JADBABAIE, A. Distributed coverage verification algorithms in sensor networks without location information. *IEEE Transactions on Automatic Control 55*, 8 (2010), to appear.

[21] TARDOS, E. A strongly polynomial minimum cost circulation algorithm. *Combinatorica 5*, 3 (September 1985), 247–255.

[22] TARDOS, E. A strongly polynomial algorithm to solve combinatorial linear programs. *Operations Research 34*, 2 (March 1986), 250–256.

[23] TRUEMPER, K. A decomposition theory for matroids. VII. analysis of minimal violation matrices. *Journal of Combinatorial Theory, Series B 55*, 2 (1992), 302–335.

[24] VEINOTT, JR., A. F., AND DANTZIG, G. B. Integral extreme points. *SIAM Review 10*, 3 (1968), 371–372.

Appendix

Boundary matrices for non-orientable surfaces

The boundary matrices $[\partial_2]$ for the Möbius strip and projective plane triangulations shown in Figure 1 are given below. The row numbers are edge numbers and the column numbers are triangle numbers which are displayed in Figure 1.

$[\partial_2]$ for Möbius strip :

$$
\begin{array}{rc}
 & \begin{array}{rrrrrr} 0: & 1: & 2: & 3: & 4: & 5: \end{array} \\
\begin{array}{r} 0:\\ 1:\\ 2:\\ 3:\\ 4:\\ 5:\\ 6:\\ 7:\\ 8:\\ 9:\\ 10:\\ 11: \end{array} &
\left[\begin{array}{rrrrrr}
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array} \tag{11}
$$

$[\partial_2]$ for projective plane :

$$
\begin{array}{rc}
 & \begin{array}{rrrrrrrrrr} 0: & 1: & 2: & 3: & 4: & 5: & 6: & 7: & 8: & 9: \end{array} \\
\begin{array}{r} 0:\\ 1:\\ 2:\\ 3:\\ 4:\\ 5:\\ 6:\\ 7:\\ 8:\\ 9:\\ 10:\\ 11:\\ 12:\\ 13:\\ 14: \end{array} &
\left[\begin{array}{rrrrrrrrrr}
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array} \tag{12}
$$